Multiplicative Hom-Lie superalgebra structures on infinite dimensional simple Lie superalgebras of vector fields

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Abstract This paper considers the multiplicative Hom-Lie superalgebra structures on infinite dimensional simple Lie superalgebras of vector fields with characteristic zero. The main result is that there is only the multiplicative Hom-Lie superalgebra structure on these Lie superalgebras.

Keywords: Lie superalgebra, Hom-Lie superalgebra structure, automorphism

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0. Introduction

Hom-Lie algebra structures were introduced and studied in [1–4]. In 2008, Q. Jin and X. Li gave a description of Hom-Lie algebra structures of Lie algebras and determined the isomorphic classes of nontrivial Hom-Lie algebra structures of finite dimensional semi-simple Lie algebras [5]. The Hom-Lie algebras have been sufficiently studied in [6, 7].

The theory of Lie superalgebras has seen a significant development. For example, V. G. Kac classified the finite dimensional simple Lie superalgebras and the infinite dimensional simple linearly compact Lie superalgebras over algebraically closed fields of characteristic zero [8, 9]. In 2010, F. Ammar and A. Makhlouf generalized Hom-Lie algebras to Hom-Lie superalgebras [10]. In 2012, B. T. Cao and L. Luo proved that there is only the trivial Hom-Lie superalgebra structure on a finite dimensional simple Lie superalgebra of characteristic zero [11].

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This paper is motivated by the results and methods relative to finite dimensional simple Lie superalgebras with characteristic zero (cf. [11]). In Section 1 the notations of infinite dimensional simple Lie superalgebras of vector fields were introduced. In Section 2 the multiplicative Hom-Lie superalgebra structures on infinite dimensional simple Lie superalgebras of vector fields were studied. We proved that there is only the trivial multiplicative Hom-Lie superalgebra structure on infinite dimensional simple Lie superalgebras of vector fields.

1. Preliminaries

Throughout $\mathbb{F}$ is a field of characteristic zero. $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ is the additive group of two elements. $\mathbb{N}$ and $\mathbb{N}_0$ are the sets of positive integers and nonnegative integers, respectively. $\mathbb{F}[x_1, \ldots, x_m]$ denotes the polynomial algebra over $\mathbb{F}$ in even indeterminates $x_1, x_2, \ldots, x_m$, where $m > 3$. For positive integers $n > 3$, let $\Lambda(n)$ be the Grassmann superalgebra over $\mathbb{F}$ in the n odd indeterminates $x_{m+1}, x_{m+2}, \ldots, x_{m+n}$. Clearly,

$$\Lambda(m,n) := \mathbb{F}[x_1, \ldots, x_m] \otimes \Lambda(n).$$

is an associative commutative superalgebra.

Let ∂_r be the superderivation of $\Lambda(m,n)$ defined by $\partial_r(x_s) = \delta_{rs}$ for $r, s \in \overline{1, m+n}$. The *generalized Witt superalgebra* $W(m,n)$ is $\mathbb{F}$-spanned by $\{f_r \partial_r \mid f_r \in \Lambda(m,n), r \in \overline{1, n+m}\}$. Note that $W(m,n)$ is a free $\Lambda(m,n)$-module with basis $\{\partial_r \mid r \in \overline{1, m+n}\}$.

For a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we write $|x| := \theta$ for the *parity* of a homogeneous element $x \in V_\theta$, $\theta \in \mathbb{Z}_2$. Once the symbol $|x|$ appears, it will imply that x is a $\mathbb{Z}_2$-homogeneous element.

The following symbols will be frequently used in this paper:

- $i'_H = i'_K := \begin{cases} i+r, & if \quad 1 \leq i \leq r \\ i-r, & if \quad r < i \leq 2r \\ i, & if \quad i \in \overline{m+1, m+n}, \end{cases} \qquad \tau(i) := \begin{cases} 1, & if \quad 1 \leq i \leq r \\ -1, & if \quad r < i \leq 2r \\ 1, & if \quad i \in \overline{m+1, m+n}, \end{cases}$
  for $m = 2r$ or $m = 2r+1$;
- $i'_X := \begin{cases} i+m, & \text{if } i \in \overline{1, m} \\ i-m, & \text{if } i \in \overline{m+1, 2m}, \end{cases}$ where $X = HO, KO, SHO$ and SKO;
- $\mathrm{div}(f_r \partial_r) = (-1)^{|\partial_r||f_r|} \partial_r(f_r)$, where div is a linear mapping from $W(m,n)$ to $\Lambda(m,n)$;
- $\mathrm{div}_\lambda(f) := (-1)^{|f|} 2 \left(\sum\limits_{i=1}^{m} \partial_i \partial_{i'_{SKO}}(f) + \left(\mathfrak{D} - m\lambda \mathrm{id}_{\Lambda(m,m+1)} \right) \partial_{2m+1}(f) \right)$, where $f \in \Lambda(m,m)$ and $\lambda \in \mathbb{F}$;
- $\mathrm{D}_{ij}(f) := (-1)^{|\partial_i||\partial_j|} \partial_i(f) \partial_j - (-1)^{(|\partial_i| + |\partial_j|)|f|} \partial_j(f) \partial_i$, where $f \in \Lambda(m,m)$;
- $\mathrm{D_H}(f) := \sum\limits_{i=1}^{m+n} \tau(i)(-1)^{|\partial_i||f|} \partial_i(f) \partial_{i'_H}$, where $m = 2r$ and $f \in \Lambda(m,m)$;
- $\mathrm{D_K}(f) := \sum\limits_{m \neq i=1}^{m+n} (-1)^{|\partial_i||f|} \left(x_i \partial_m(f) + \tau(i'_K) \partial_{i'_K}(f) \right) \partial_i$
  $+ \Big(2f - \sum\limits_{m \neq i=1}^{m+n} x_i \partial_i(f) \Big) \partial_m$, where $m = 2r+1$ and $f \in \Lambda(m,m)$;

- $\mathrm{D_{HO}}(f) := \sum_{i=1}^{2m}(-1)^{|\partial_i||f|}\partial_i(a)\partial_{i'_{HO}}$, where $f \in \Lambda(m,m)$;
- $\mathrm{D_{KO}}(f) := \mathrm{D_{HO}}(f) + (-1)^{|a|}\partial_{2m+1}(f)\mathfrak{D} + (\mathfrak{D}(f) - 2f)\partial_{2m+1}, \quad \mathfrak{D} := \sum_{i=1}^{2m} x_i\partial_i$, where $f \in \Lambda(m,m)$;
- $\nu := \delta_{X,K}m + \delta_{X,KO}(2m+1) + \delta_{X,SKO}(2m+1)$.

The following infinite dimensional Lie superalgebras of vector fields, which are the simple Lie superalgebra contained in $W(m,n)$, are defined as follows (cf. [9]):

- $S\,(m,n) := \mathrm{span}_{\mathbb{F}}\left\{\mathrm{D_{ij}}(f) \mid f \in \Lambda\,(m,n)\,, i,j \in \overline{1, m+n}\right\};$
- $H\,(m,n) := \{\mathrm{D_H}(f) \mid f \in \Lambda\,(m,n)\}\,;$
- $K\,(m,n) := \{\mathrm{D_K}(f) \mid f \in \Lambda\,(m,n)\}\,;$
- $HO\,(m,m) := \{\mathrm{D_{HO}}(f) \,|\, f \in \Lambda(m,m)\};$
- $KO(m,m+1) := \{\mathrm{D_{KO}}(f) \mid f \in \Lambda(m,m+1)\};$
- $SHO(m,m) := [SHO'(m,m), SHO'(m,m)]$, where $SHO'(m,m) := \{D \in HO(m,m) \mid \mathrm{div}(D) = 0\};$
- $SKO(m,m+1;\lambda) := [SKO'(m,m+1;\lambda), SKO'(m,m+1;\lambda)]$, where $SKO'(m,m+1;\lambda) := \{\mathrm{D_{KO}}(f) \mid \mathrm{div}_\lambda(f) = 0, f \in \Lambda(m,m+1)\}$ and $\lambda \in \mathbb{F}$.

Hereafter, write X for W, S, H, K, HO, KO, SHO or SKO.

2. Multiplicative Hom-Lie superalgebra

Definition 2.1. *A multiplicative Hom-Lie superalgebra is a triple* $(\mathfrak{g}, [,], \sigma)$ *consisting of a* $\mathbb{Z}_2$*-graded vector space* $\mathfrak{g}$*, a bilinear map* $[,] : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ *and an even linear map* $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ *satisfying*

$$\sigma[x,y] = [\sigma(x), \sigma(y)], \tag{2.1}$$

$$[x,y] = -(-1)^{|x||y|}[y,x],$$

$$(-1)^{|x||z|}[\sigma(x),[y,z]] + (-1)^{|y||x|}[\sigma(y),[z,x]] + (-1)^{|z||y|}[\sigma(z),[x,y]] = 0, \tag{2.2}$$

where x, y *and* z *are homogeneous elements in* $\mathfrak{g}$.

For any simple Lie superalgebra $\mathfrak{g}$, denote its Lie bracket by $[,]$ and take an even linear map $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$. We say $(\mathfrak{g}, \sigma)$ is a multiplicative Hom-Lie superalgebra structure over the Lie superalgebra $\mathfrak{g}$ if $(\mathfrak{g}, [,], \sigma)$ is a multiplicative Hom-Lie superalgebra. Suppose $\sigma \neq 0$. Eq. (2.1) and the simplicity of $\mathfrak{g}$ show that σ is a monomorphism of $\mathfrak{g}$. In particularly, if $\sigma = \mathrm{id}$ or $\sigma = 0$, the multiplicative Hom-Lie superalgebra structure is called trivial. Before consider the multiplicative Hom-Lie superalgebra structures on $X(m,n)$, we introduce the gradations on them as in [9]. For any $(m+n)$-tuple $\underline{\alpha} := (\alpha_1, \ldots, \alpha_m \mid \alpha_{m+1}, \ldots, \alpha_{m+n}) \in \mathbb{N}^{m+n}$, we may define a gradation on $W(m,n)$ by letting $\mathrm{deg}x_i := \alpha_i =: -\mathrm{deg}\partial_i$, where $i \in \overline{1, m+n}$. Thus $W(m,n)$ becomes a graded Lie superalgebra of finite depth, i.e., we have

$$W(m,n) = \oplus_{j=-h}^{\infty} W(m,n)_{\underline{\alpha},[j]},$$

where h is a positive integer. Put

$$\underline{\gamma} := \underline{1} + \delta_{X,K}\varepsilon_m + \delta_{X,KO}\varepsilon_{2m+1} + \delta_{X,SKO}\varepsilon_{2m+1} \in \mathbb{N}^{m+n}$$

and sometimes omit the subscript $\underline{\gamma}$. Putting

$$X(m,n)_{\underline{\gamma},[i]} := X(m,n) \cap W(m,n)_{\underline{\gamma},[i]},$$

one sees that $X(m,n)$ is graded by $(X(m,n)_{\underline{\gamma},[i]})_{i\in\mathbb{Z}}$. In particular,

- $X(m,n)_{[-2]} = \mathbb{F}\cdot \mathrm{D_X}(1)$, where $X := K, KO$ or SKO;
- $X(m,n)_{[-1]} = \mathrm{span}_{\mathbb{F}}\{\partial_i \mid i \in \overline{1,m+n}\}$, where $X := W$ or S;
- $X(m,n)_{[-1]} = \mathrm{span}_{\mathbb{F}}\{\mathrm{D_X}(x_i) \mid \nu \neq i \in \overline{1,2n}\}$, where $X := H, K, HO, KO, SHO$ or SKO;
- $W(m,n)_{[0]} = \mathrm{span}_{\mathbb{F}}\{x_i\partial_j \mid i,j \in \overline{1,m+n}\}$;
- $S(m,n)_{[0]} = \mathrm{span}_{\mathbb{F}}\{x_i\partial_j, x_i\partial_i - x_j\partial_j \mid i \neq j \in \overline{1,m+n}\}$;
- $X(m,n)_{[0]} = \mathrm{span}_{\mathbb{F}}\{\mathrm{D_H}(x_ix_j), \delta_{X,K}\mathrm{D_H}(x_m), \delta_{X,KO}\mathrm{D_H}(x_{2m+1}) \mid i,j \in \overline{1,m+n}\}$, where $X := H, K, HO$ or KO;
- $X(m,n)_{[0]} = \mathrm{span}_{\mathbb{F}}\{\mathrm{D_X}(x_ix_j), \mathrm{D_X}(x_ix_{i'} - x_jx_{j'}), \mathrm{D_X}(x_{2n+1} + \delta_{X,SKO}n\lambda x_ix_{i'}) \mid i \neq j' \in \overline{1,m+n}\}$, where $X := SHO$ or SKO.

Next we give an equation and several lemmas needed in the sequel. The verifications are straightforward. The equation will be used without notice: for $f, g \in \Lambda(m,n)$,

$$[\mathrm{D_X}(f), \mathrm{D_X}(g)] = \mathrm{D_X}\left(\mathrm{D_X}(f)(g) - 2\left(\delta_{X,K} - (-1)^{|f|}\delta_{X,KO}\right)\partial_\nu(f)\, g\right).$$

Lemma 2.2. *(cf. [9]) The $\mathbb{Z}$-graded Lie superalgebra $X(m,n)$ is transitive, that is, if $x \in \mathfrak{g}_i$ with $i \geq 0$ and $[x, \mathfrak{g}_{[-1]}] = 0$, then $x = 0$.*

Lemma 2.3. *For $\mathbb{Z}$-graded Lie superalgebra $X(m,n)$, we have*

$$\mathrm{Ker}(\mathrm{ad}\partial_i) \cap X(m,n)_{[0]} = \mathrm{span}_{\mathbb{F}}\{x_j\partial_k \mid j,k \in \overline{1,m+n}, i \neq j\} \cap X(m,n)_{[0]}$$

and

$$[\mathrm{Ker}(\mathrm{ad}\partial_i) \cap X(m,n)_{[0]}, \mathrm{Ker}(\mathrm{ad}\partial_i) \cap X(m,n)_{[0]}] = \mathrm{Ker}(\mathrm{ad}\partial_i) \cap X(m,n)_{[0]},$$

where $i \in \overline{1,m+n}\backslash\nu$.

Lemma 2.4. *For $i,j,k,l \in \overline{1,m+n}$, we have that*

$$x_k\partial_l \in [\mathrm{Ker}(\mathrm{ad}x_i\partial_j) \cap X(m,n)_{[0]}, \mathrm{Ker}(\mathrm{ad}x_i\partial_j) \cap X(m,n)_{[0]}],$$

where $k \neq j, l$;

$$\mathrm{D_X}(x_kx_l) \in [\mathrm{Ker}(\mathrm{ad}\mathrm{D_X}(x_ix_j)) \cap X(m,n)_{[0]}, \mathrm{Ker}(\mathrm{ad}\mathrm{D_X}(x_ix_j)) \cap X(m,n)_{[0]}],$$

where $k \neq l \in \overline{1,m+n}\backslash\nu$, $k,l \neq i'_X, j'_X$.

The next proposition is essential for the main result in this paper.

Proposition 2.5. *If $(X(m,n),\sigma)$ is a multiplicative Hom-Lie superalgebra structure and $\sigma \neq 0$, then*

$$\sigma \mid_{X(m,n)_{[-1]}} = \mathrm{id} \mid_{X(m,n)_{[-1]}} .$$

Proof. **Case 1**: $X := W$ or S. By Eq. (2.2), we have

$$\begin{aligned} 0 &= (-1)^{|\partial_i||x_j\partial_k|}[\sigma(\partial_i),[\partial_j,x_j\partial_k]] + (-1)^{|\partial_i||\partial_j|}[\sigma(\partial_j),[x_j\partial_k,\partial_i]] \\ &\quad + (-1)^{|\partial_j||x_j\partial_k|}[\sigma(x_j\partial_k),[\partial_i,\partial_j]] \\ &= (-1)^{|\partial_i||x_j\partial_k|}[\sigma(\partial_i),\partial_k], \end{aligned}$$

where $j \neq i, k \in \overline{1, m+n}$. By Lemma 2.2, we have

$$\sigma(X(m,n)_{[-1]}) = X(m,n)_{[-1]}.$$

Then for any $i \in \overline{1, m+n}$, one may suppose $\sigma(\partial_i) = \sum\limits_{l=1}^{m+n} a_l\partial_l$, where $a_l \in \mathbb{F}$. Lemma 2.3 and Eq. (2.2) imply that $\sigma(\partial_i) = a_i\partial_i$. For distinct $i, j, k \in \overline{1, m+n}$, put $x = x_j\partial_j - x_k\partial_k$, $y = \partial_i$ and $z = x_i\partial_j$. Then Eq. (2.2) implies that

$$[\sigma(x_j\partial_j - x_k\partial_k),\partial_j] + [\sigma(\partial_i),x_i\partial_j] = 0. \tag{2.3}$$

Suppose σ^{-1} is an left linear inverse of σ (vector space). Then

$$\sigma^{-1}([\sigma(x_j\partial_j - x_k\partial_k),\partial_j]) = [x_j\partial_j - x_k\partial_k,\sigma^{-1}(\partial_j)] = [x_j\partial_j - x_k\partial_k, a_j^{-1}\partial_j] = -a_j^{-1}\partial_j.$$

Hence

$$[\sigma(x_j\partial_j - x_k\partial_k),\partial_j] = -\partial_j.$$

By Eq. (2.3), we have $a_i = 1$, where $i \in \overline{1, m+n}$. That is

$$(\sigma - \mathrm{id}) \mid_{X(m,n)_{[-1]}} = 0.$$

Case 2: $X := H, K, HO, KO, SHO$ or SKO. For $i, j, k \in \overline{1, m+n}\backslash\nu$, Eq. (2.2) implies that

$$\begin{aligned} 0 &= (-1)^{|\mathrm{D_X}(x_i)||\mathrm{D_X}(x_{j'_X}x_{k'_X})|}[\sigma(\mathrm{D_X}(x_i)),[\mathrm{D_X}(x_j),\mathrm{D_X}(x_{j'_X}x_{k'_X})]] \\ &\quad + (-1)^{|\mathrm{D_X}(x_j)||\mathrm{D_X}(x_i)|}[\sigma(\mathrm{D_X}(x_j)),[\mathrm{D_X}(x_{j'_X}x_{k'_X}),\mathrm{D_X}(x_i)]] \\ &\quad + (-1)^{|\mathrm{D_X}(x_{j'_X}x_{k'_X})||\mathrm{D_X}(x_j)|}[\sigma(\mathrm{D_X}(x_{j'_X}x_{k'_X})),[\mathrm{D_X}(x_i),\mathrm{D_X}(x_j)]]. \end{aligned}$$

It follows that

$$[\sigma(\mathrm{D_X}(x_i)),\mathrm{D_X}(x_{k'_X})] = 0, \quad i \neq j, j'_X, k \tag{2.4}$$

and

$$[\sigma(\mathrm{D_X}(x_i)),\mathrm{D_X}(x_{i'_X})] = [\sigma(\mathrm{D_X}(x_j)),\mathrm{D_X}(x_{j'_X})], \quad i \neq j, j'. \tag{2.5}$$

By Eq. (2.4), (2.5) and Lemma 2.2, it is easy to obtain that

$$\sigma(\mathrm{D_X}(x_i)) = a_i\mathrm{D_X}(1) + \sum_{\nu\neq l=1}^{m+n} a_{il}\mathrm{D_X}(x_l)$$

for some $a_i, a_{il} \in \mathbb{F}$.

Subcase 2.1: $X := H, HO$, or SHO. Lemma 2.3 and Eq. (2.2) imply that $a_{ik} = 0$ for all $i \neq k \in \overline{1, m+n} \backslash \nu$. Thus,

$$(\sigma - \mathrm{id}) \mid_{X(m,n)_{[-1]}} = 0.$$

Subcase 2.2: $X := K, KO$ or SKO. Put $x = \mathrm{D_X}(x_i)$, $y = \mathrm{D_X}(x_j)$ and

$$z = \mathrm{D_X}(x_{j'_X} x_\nu + \delta_{X,SKO}(-1)^{|x_{j'_X}|}(m\lambda - 1)x_k x_{k'_X} x_{j'_X}).$$

Eq. (2.2) implies that

$$\begin{aligned} 0 \;\; = \;\; & (-1)^{|\mathrm{D_X}(x_i))||z|}[\sigma(\mathrm{D_X}(x_i)), \mathrm{D_X}(x_\nu + \delta_{X,SKO}(m\lambda - 1)x_k x_{k'_X} + x_j x_{j'_X})] \\ & +(-1)^{|\mathrm{D_X}(x_i)||\mathrm{D_X}(x_j)|}[\sigma(\mathrm{D_X}(x_j)), \mathrm{D_X}(x_i x_{j'_X})]. \end{aligned}$$

Hence $a_i = 0$. Take $i, j, k \in \overline{1, m+n} \backslash \nu$ and $i \neq j, j'_X, k, k'_X$. Put $x = \mathrm{D_X}(x_j x_k) \in X(m,n)_{\bar{0}}$, $y = \mathrm{D_X}(x_i)$ and $z = \mathrm{D_X}(x_{i'_X} x_{j'_X})$. By Eq. (2.2) again, we have

$$\begin{aligned} 0 \;\; = \;\; & (-1)^{|\partial_i||x_i|}[\sigma(\mathrm{D_X}(x_j x_k)), \mathrm{D_X}(x_{j'_X})] \\ & +(-1)^{|\partial_{j'_X}||x_{i'_X} x_{j'_X}| + |\partial_{j'_X}||x_{i'_X}|}[\sigma(\mathrm{D_X}(x_i)), \mathrm{D_X}(x_{i'_X} x_k)]. \end{aligned}$$

Suppose σ^{-1} is a left inverse of σ. Then

$$\begin{aligned} \sigma^{-1}([\sigma(\mathrm{D_X}(x_j x_k)), \mathrm{D_X}(x_{j'_X})]) &= [\mathrm{D_X}(x_j x_k), \sigma^{-1}(\mathrm{D_X}(x_{j'_X}))] \\ &= [\mathrm{D_X}(x_j x_k), a^{-1}_{j'_X j'_X} \mathrm{D_X}(x_{j'_X})] \\ &= -(-1)^{|\partial_{j'_X}||x_{j'_X}|} a^{-1}_{j'_X j'_X} \mathrm{D_X}(x_k). \end{aligned}$$

Hence

$$\begin{aligned} -(-1)^{|\partial_{j'_X}||x_{j'_X}|} a_{kk} a^{-1}_{j'_X j'_X} \mathrm{D_X}(x_k) &= [\sigma(\mathrm{D_X}(x_j x_k)), \mathrm{D_X}(x_{j'_X})] \\ &= -(-1)^{|\partial_{j'_X}||x_{j'_X}| + |\partial_i||x_i|}[\sigma(\mathrm{D_X}(x_i)), \mathrm{D_X}(x_{i'_X} x_k)] \\ &= -(-1)^{|\partial_{j'_X}||x_{j'_X}|} a_{ii} \mathrm{D_X}(x_k). \end{aligned}$$

The arbitrariness of j and k implies that $a_{ii} = 1$. Thus,

$$(\sigma - \mathrm{id}) \mid_{X(m,n)_{[-1]}} = 0.$$

□

Proposition 2.6. *If $(X(m,n), \sigma)$ is a multiplicative Hom-Lie superalgebra structure and $\sigma \neq 0$, then*

$$\sigma \mid_{X(m,n)_{[0]}} = \mathrm{id} \mid_{X(m,n)_{[0]}} .$$

Proof. **Case 1**: $X := W$ or S. Put $x \in X(m,n)_{[0]}$. Then by Proposition 2.5, we have

$$[\sigma(x), \partial_i] = [\sigma(x), \sigma(\partial_i)] = \sigma([x, \partial_i]) = [x, \partial_i]$$

for all $i \in \overline{1, m+n}$. By Lemma 2.2, we may write

$$\sigma(x_i\partial_j) = x_i\partial_j + \sum_{s=1}^{m+n} a_{ijs}\partial_s,$$

where $a_{ijs} \in \mathbb{F}$. By Lemma 2.4 and Eq. (2.2), we have

$$a_{ijk}\partial_l = [\sum_{s=1}^{m+n} a_{ijs}\partial_s, x_k\partial_l] = 0$$

for $k, l \in \overline{1, m+n}$ and $k \neq j, l$. By the arbitrariness of l, we know $a_{ijk} = 0$ for all $j \neq k \in \overline{1, m+n}$. Put $x = x_i\partial_j$, $y = x_j\partial_l$, $z = x_l\partial_l - (-1)^{(|x_l|+|x_s|)}x_s\partial_s$. By Eq. (2.2) we have that

$$[\sigma(x_i\partial_j), x_j\partial_l] + [\sigma(x_l\partial_l - (-1)^{(|x_l|+|x_s|)}x_s\partial_s), [x_i\partial_j, x_j\partial_l]] = 0.$$

Furthermore,

$$[a_{ijj}\partial_j, x_j\partial_l] + [a_{lll}\partial_l - a_{sss}\partial_s, x_i\partial_l] = 0.$$

Then $a_{ijj} = 0$. Summarizing, we have $\sigma(x_i\partial_j) = x_i\partial_j$.

Case 2: $X := H, K, HO, KO, SHO$ or SKO. For $x \in X(m,n)_{[0]}$, by Proposition 2.5, we have

$$[\sigma(x), \mathrm{D_X}(x_i)] = [\sigma(x), \sigma(\mathrm{D_X}(x_i))] = \sigma([x, \mathrm{D_X}(x_i)]) = [x, \mathrm{D_X}(x_i)] \tag{2.6}$$

where $i \in \overline{1, m+n}\backslash\nu$. By Lemma 2.2, we may write

$$\sigma(\mathrm{D_X}(x_ix_j)) = \mathrm{D_X}(x_ix_j) + \sum_{\nu\neq s=1}^{m+n} a_{ijs}\mathrm{D_X}(x_s) + a_{ij}\mathrm{D_X}(1),$$

where $\mathrm{D_X}(x_ix_j) \in X(m,n)_{[0]}$ and $a_{ij}, a_{ijs} \in \mathbb{F}$. By Lemma 2.4 and Eq. (2.2) we have that

$$\pm a_{ijk'_X}\mathrm{D_X}(x_l) \pm a_{ijl'_X}\mathrm{D_X}(x_k) = [\sum_{s=1}^{m+n} a_{ijs}\mathrm{D_X}(x_s), \mathrm{D_X}(x_kx_l)] = 0,$$

where $k \neq l \in \overline{1, m+n}\backslash\nu$ and $k, l \neq i'_X, j'_X$. That is $a_{ijs} = 0$ for all $i, j \neq s \in \overline{1, m+n}\backslash\nu$. Then

$$\sigma(\mathrm{D_X}(x_ix_j)) = \mathrm{D_X}(x_ix_j) + a_{iji}\mathrm{D_X}(x_i) + a_{ijj}\mathrm{D_X}(x_j) + a_{ij}\mathrm{D_X}(1).$$

If $k = i'_X$ or $k = j'_X$, by Eq. (2.2) we have

$$\pm a_{ijk}\mathrm{D_X}(x_l) = [a_{iji}\mathrm{D_X}(x_i) + a_{ijj}\mathrm{D_X}(x_j), \mathrm{D_X}(x_kx_l)] = 0$$

Hence

$$\sigma(\mathrm{D_X}(x_ix_j)) = \mathrm{D_X}(x_ix_j) + a_{ij}\mathrm{D_X}(1). \tag{2.7}$$

Subcase 2.1: $X := H$ or HO. From Eq. (2.7), we have

$$(\sigma - \mathrm{id}) \mid_{X(m,n)_{[0]}} = 0.$$

Subcase 2.2: $X := SHO$. By Eq. (2.7) again, for $i \neq j \in \overline{1,m} \backslash \nu$ we can obtain

$$\begin{aligned}\sigma(\mathrm{D_X}(x_i x_{i'_X} - x_j x_{j'_X})) &= \sigma([\mathrm{D_X}(x_i x_j), \mathrm{D_X}(x_{i'_X} x_{j'_X})]) \\ &= [\sigma(\mathrm{D_X}(x_i x_j)), \sigma(\mathrm{D_X}(x_{i'_X} x_{j'_X}))] \\ &= [\mathrm{D_X}(x_i x_j), \mathrm{D_X}(x_{i'_X} x_{j'_X})] = \mathrm{D_X}(x_i x_{i'_X} - x_j x_{j'_X}).\end{aligned} \tag{2.8}$$

From Eq. (2.7) and (2.8), we know

$$(\sigma - \mathrm{id}) \mid_{SHO(m,n)_{[0]}} = 0.$$

Subcase 2.3: $X := K, KO$ or SKO. Take $x = \mathrm{D_X}(x_i x_j)$, $y = \mathrm{D_X}(x_k)$ and $z = \mathrm{D_X}(x_{k'_X} x_\nu$ $+(-1)^{|x_{k'_X}|}(n\lambda - 1)x_l x_{l'_X})$, where $i, j, k, k'_X, l, l'_X \in \overline{1, m+n}$ are distinct. By Eq. (2.2), we have

$$\begin{aligned}-2a_{ij} = [\sigma(x), [y, z]] \;=\; & (-1)^{|\partial_k||x_k|}[\mathrm{D_X}(x_i x_j) + a_{ij}\mathrm{D_X}(1), \\ & \mathrm{D_X}(x_\nu + (-1)^{|x_{k'_X}|}(n\lambda - 1)x_l x_{l'_X} + (-1)^{|x_{k'_X}|}x_k x_{k'_X})] = 0.\end{aligned}$$

Hence

$$\sigma(\mathrm{D_X}(x_i x_j)) = \mathrm{D_X}(x_i x_j).$$

By Eq. (2.6), we can write

$$\begin{aligned}\sigma(\mathrm{D_X}(x_\nu + \delta_{X,SKO} n\lambda x_j x_{j'_X})) &= \mathrm{D_X}(x_\nu + \delta_{X,SKO} n\lambda x_j x_{j'_X}) \\ &\quad + \sum_{\nu \neq s=1}^{m+n} a_{\nu js}\mathrm{D_X}(x_s) + a_{\nu j}\mathrm{D_X}(1).\end{aligned} \tag{2.9}$$

Using the method above, we can obtain easily

$$\sigma(\mathrm{D_X}(x_\nu + \delta_{X,SKO} n\lambda x_j x_{j'_X})) = \mathrm{D_X}(x_\nu + \delta_{X,SKO} n\lambda x_j x_{j'_X}).$$

By Eq. (2.7), (2.8) and (2.9), we have

$$(\sigma - \mathrm{id}) \mid_{X(m,n)_{[0]}} = 0.$$

The proof is complete. □

Theorem 2.7. *There is only the trivial multiplicative Hom-Lie superalgebra structure on the infinite dimensional simple Lie superalgebras of vector fields.*

Proof. Let $(X(m,n), \sigma)$ be a multiplicative Hom-Lie superalgebra structure and $\sigma \neq 0$. By Propositions 2.5 and 2.6, we have

$$\sigma \mid_{X(m,n)_{[-1]} \oplus X(m,n)_{[0]}} = \mathrm{id} \mid_{X(m,n)_{[-1]} \oplus X(m,n)_{[0]}} .$$

Now let $x \in X(m,n)_{[l]}$ and $y, z \in X(m,n)_{[-1]} \oplus X(m,n)_{[0]}$, where $l \geq 1$. By Eq. (2.2), we have

$$[\sigma(x) - x, [y, z]] = 0.$$

Then $\sigma(x) - x = 0$. We get $\sigma = \mathrm{id}$. The proof is complete. □

References

[1] J. T. Hartwig, D. Larsson, S. D. Silvesrov. Derformations of the Lie algebras using σ-derivations. *J. Algebra,* **295** (2007): 314–361.

[2] D. Larsson, S. D. Silvesrov. Quasi-hom-Lie algebras, Central extensions and 2-cocycle-like identities. *J. Algebra,* **288** (2005): 321–344.

[3] D. Larsson, S. D. Silvesrov. Quasi-derformations of $sl_2(\mathbb{F})$ using twisted derivations. *Comm. Algebra,* **35** (2007): 4303–4318.

[4] D. Larsson, S. D. Silvesrov. Graded quasi-Lie algebras. *Czechoslovak J. Phys.,* **55** (2005): 1473–1478.

[5] Q. Jin, X. Li. Hom-Lie algebra stuctures on semi-simple Lie algebras. *J. Algebra,* **319** (2008): 1398–1408.

[6] S. Benayadi, A. Makhlouf. Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms. *arXiv: 1009.4226,* (2010).

[7] Y. H. Sheng. Representations of Hom-Lie algebras. *Algebr. Represent. Theory,* DOI: 10.1007/s10468-011-9280-8.

[8] V. G. Kac. Lie superalgebras. *Adv. Math.,* **26** (1977): 8–96.

[9] V. G. Kac. Classification of infinite dimensional simple linearly compact Lie superalgebras. *Adv. Math.,* **139** (1998): 1–55.

[10] F. Ammar, A. Makhlouf. Hom-Lie superalgebras and Hom-Lie admissible superalgebras. *J. Algebra,* **324** (2010): 1513–1528.

[11] B. T. Cao, L. Luo. Hom-Lie superalgebra structures on finite-dimensional simple Lie superalgebras. *arXiv:1203.0136* (2012).